

Majorana spinors and extended Lorentz symmetry in four-dimensional theory

V. I. Tselyaev

*Nuclear Physics Department, V. A. Fock Institute of Physics,
St. Petersburg State University, 198504, St. Petersburg, Russia**

(Dated: February 2, 2008)

Abstract

An extended local Lorentz symmetry in four-dimensional (4D) theory is considered. A source of this symmetry is a group of general linear transformations of four-component Majorana spinors $GL(4, M)$ which is isomorphic to $GL(4, R)$ and is the covering of an extended Lorentz group in a 6D Minkowski space $M(3, 3)$ including superluminal and scaling transformations. Physical space-time is assumed to be a 4D pseudo-Riemannian manifold. To connect the extended Lorentz symmetry in the $M(3, 3)$ space with the physical space-time, a fiber bundle over the 4D manifold is introduced with $M(3, 3)$ as a typical fiber. The action is constructed which is invariant with respect to both general 4D coordinate and local $GL(4, M)$ spinor transformations. The components of the metric on the 6D fiber are expressed in terms of the 4D pseudo-Riemannian metric and two extra complex fields: 4D vector and scalar ones. These extra fields describe in the general case massive particles interacting with an extra $U(1)$ gauge field and weakly interacting with ordinary particles, i.e. possessing properties of invisible (dark) matter.

PACS numbers: 03.30.+p, 04.90.+e, 95.35.+d

*E-mail address: tselyaev@nuclpc1.phys.spbu.ru

I. INTRODUCTION

Possible modification of the usual Lorentz symmetry (from the standpoint of its extension or, on the contrary, breakdown) is of obvious interest since it underlies many modern physical theories. Perhaps, one of the most intriguing questions is an extension of the Lorentz symmetry admitting superluminal transformations. These transformations can be introduced immediately in a flat two-dimensional (2D) space-time (see [1]). In this case the generalized Lorentz transformation between two frames with space-time coordinates x, t and x', t' has the form (in units where $c = 1$):

$$x' = (x - vt)/\sqrt{|1 - v^2|}, \quad t' = (t - vx)/\sqrt{|1 - v^2|} \quad (1)$$

where v is the relative velocity of the frames. The absolute value of the quadratic form $t^2 - x^2$ is invariant under this transformation at any $v^2 \neq 1$ but its sign is changed at $v^2 > 1$ (superluminal case). In the limit $v^2 \rightarrow \infty$ one obtains: $x' \rightarrow \pm t$, $t' \rightarrow \pm x$. However, if one starts with the 4D Minkowski space $M(3, 1)$, situation becomes more complicated. Usually, in this case two ways are considered. In the first approach, imaginary quantities are introduced in the 4D or 3D space-time in addition to or instead of the usual real coordinates (see Refs. [2, 3, 4, 5, 6] and references therein for more details). In the second approach, the 4D Minkowski space is replaced by the 6D one $M(3, 3)$ by introducing two extra time-like coordinates, but all the space-time coordinates remain real. In the $M(3, 3)$ space, the superluminal transformations are the combinations of the 6D rotations, reflections, and the space-time permutation of the form

$$x_1 \rightleftharpoons t_1, \quad x_2 \rightleftharpoons t_2, \quad x_3 \rightleftharpoons t_3. \quad (2)$$

This approach was proposed in Refs. [3, 4], developed in Refs. [6, 7, 8, 9] (see also references therein), and further elaborated and applied in Refs. [10, 11, 12]. However, both these methods face a question how to reconcile additional coordinate variables (imaginary or real ones) with the observed four-dimensionality of the physical space-time. Possible ways to overcome this difficulty immediately within the framework of the above-mentioned approaches were discussed in the papers cited above (see also their discussion and criticism in Refs. [13, 14, 15]). In the present paper, the following solution of this problem is proposed: physical space-time is regarded as a 4D pseudo-Riemannian manifold which is a base of a vector bundle with the $M(3, 3)$ space as a typical fiber. The superluminal transformations concern only the fiber of the frame bundle constructed from this vector bundle.

Another question arising in these approaches is how to introduce spinor fields in the theory with an extended Lorentz symmetry. In Refs. [16, 17], a spinor representation of the extended Lorentz group in the $M(3, 3)$ space was introduced with the help of eight-component Dirac spinors. Corresponding Dirac equations were considered and their properties and properties of their solutions were analyzed. However, there remains a question of the interpretation of additional spinor components in this representation. As a matter of fact, there is no need

to complicate the situation. The problem is simplified if we choose as a starting point not the properties of the space-time transformations but the properties of the spinors. More specifically, we will consider a group of general linear transformations of four-component Majorana spinors. This group is the covering of an extended Lorentz group in the $M(3, 3)$ space which includes superluminal and scaling transformations. Thus, the extension of the usual Lorentz symmetry is introduced in a natural way. On the other hand, Majorana spinors are very simple objects which can be used to construct usual four-component chiral spinors without introducing any additional components.

In the present paper, the action which includes Majorana spinor fields and obeys the extended symmetry conditions is considered. It will be shown that its spinor part reduces to the spinor action of the standard 4D theory under certain gauge conditions. Nevertheless, the action obtained in this approach contains additional bosonic fields which do not enter the standard theory. These fields describe in the general case massive particles possessing, under certain conditions, the properties of the invisible (dark) matter. Nowadays, there are numerous arguments in favour of the existence of the dark matter (DM). At the same time, its structure is unknown so far, though there are variety of models predicting its existence and particle content. A review of the current status of the DM problem can be found, e.g., in Refs. [18, 19]. Here we consider only the simplest possibility to apply the present approach to this problem.

The paper is organized as follows. In Sec. II an outline of the extended Lorentz group in the $M(3, 3)$ space is given. The group of general linear transformations of four-component Majorana spinors is described in Sec. III. In Sec. IV the connection between the extended Lorentz symmetry and the physical space-time is considered. The vector bundle and the frame bundle over the 4D manifold are introduced. In Sec. V the local gauge transformations are considered and the spin connections are determined. The chiral spinor fields with the extended symmetry are introduced in Sec. VI. In Sec. VII the spinor action is constructed and the discrete symmetries are considered. The extra fields arising in the theory and their interpretation as the dark matter candidates are discussed in Sec. VIII. A summary is given in the last Section. Appendices contain notations, conventions, auxiliary formulas, and proof of some equations used in the paper.

II. EXTENDED LORENTZ GROUP IN SIX-DIMENSIONAL SPACE

We start with a brief description of an extended Lorentz group $EL(3, 3)$ which is defined in this paper as a group of all linear transformations of the type $V'^K = \Lambda_L^K V^L$. In this equation, V^L is a vector in the 6D Minkowski space $M(3, 3)$ with a metric η_{KL} defined in Eq. (A1) (here and in the following we use notations and conventions drawn in Appendix A) and Λ_L^K is a real matrix satisfying condition

$$\Lambda_K^{K'} \eta_{K'L'} \Lambda_L^{L'} = \varkappa \eta_{KL}, \quad \varkappa \neq 0. \quad (3)$$

In the case $\varkappa = \pm 1$, these transformations were analyzed in Refs. [7, 8, 9]. Here we will consider the general case $\varkappa \neq 0$ and supplement the analysis of [7, 8, 9] with some details. Let us represent the matrices $\Lambda_{K'}^K$ and η_{KL} in the form

$$\Lambda = \begin{pmatrix} \Lambda_{s'}^s & \Lambda_{t'}^s \\ \Lambda_{s'}^t & \Lambda_{t'}^t \end{pmatrix}, \quad \eta = \begin{pmatrix} -I_3 & 0 \\ 0 & I_3 \end{pmatrix}, \quad (4)$$

where blocks $\Lambda_{s'}^s$, $\Lambda_{t'}^s$, $\Lambda_{s'}^t$, and $\Lambda_{t'}^t$ are 3×3 matrices and I_3 is the 3×3 identity matrix. Let us first consider two cases.

(i) Subluminal transformations without scaling: $\varkappa = +1$. These transformations form the group $L(3, 3) = O(3, 3)$ for which Eq. (3) reduces to $\Lambda^T \eta \Lambda = \eta$. (Here and throughout the paper Λ^T denotes a transposed matrix.) One can show (see Appendix B) that in this case the following relations hold:

$$|\det(\Lambda_{s'}^s)| = |\det(\Lambda_{t'}^t)| \geq 1, \quad (5)$$

$$\det(\Lambda) = \det(\Lambda_{s'}^s) / \det(\Lambda_{t'}^t). \quad (6)$$

Analogously to the case of the usual 4D Lorentz group, this means that the group $L(3, 3)$ consists of the following four components:

$$\left. \begin{aligned} L_+^\uparrow(3, 3) &: \det(\Lambda) = +1, \det(\Lambda_{t'}^t) \geq +1, \\ L_-^\uparrow(3, 3) &: \det(\Lambda) = -1, \det(\Lambda_{t'}^t) \geq +1, \\ L_+^\downarrow(3, 3) &: \det(\Lambda) = +1, \det(\Lambda_{t'}^t) \leq -1, \\ L_-^\downarrow(3, 3) &: \det(\Lambda) = -1, \det(\Lambda_{t'}^t) \leq -1, \end{aligned} \right\} \quad (7)$$

where $L_+^\uparrow(3, 3)$ is the proper orthochronous Lorentz group. Let us introduce two subgroups of the group $L(3, 3)$, namely: $R_{ST} = \{I_6, \mathcal{PT}\}$ and $R_S = \{I_6, \mathcal{P}\}$ where I_6 is the 6×6 identity matrix, the matrix $\mathcal{PT} = -I_6$ represents space-time reflection, and \mathcal{P} is a matrix of the space reflection:

$$\mathcal{P}_L^K = \text{diag}\{-1, -1, -1, +1, +1, +1\}. \quad (8)$$

The space reflection does not commute with arbitrary transformation $\Lambda \in L_+^\uparrow(3, 3)$. But $\mathcal{P} \Lambda \mathcal{P}^{-1} \in L_+^\uparrow(3, 3)$ if $\Lambda \in L_+^\uparrow(3, 3)$. From this it follows that the group $L(3, 3)$ can be represented as a semidirect product: $L(3, 3) = L_+^\uparrow(3, 3) \rtimes (R_S \times R_{ST})$.

(ii) Superluminal transformations without scaling: $\varkappa = -1$. Let us introduce subgroup $P_{ST} = \{I_6, \bar{I}_6\}$ where \bar{I}_6 is a matrix of the space-time permutation:

$$\bar{I}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

We have: $\bar{I}_6^2 = I_6$, $\bar{I}_6^T \eta \bar{I}_6 = -\eta$, therefore P_{ST} is a group and $P_{ST} \subset EL(3, 3)$. Further, if the equality $\bar{\Lambda}^T \eta \bar{\Lambda} = -\eta$ holds for some real matrix $\bar{\Lambda}$ and if $\Lambda = \bar{I}_6 \bar{\Lambda}$ or $\Lambda = \bar{\Lambda} \bar{I}_6$ then

we obtain: $\Lambda^T \eta \Lambda = \eta$. Consequently, any superluminal transformation without scaling can be represented in the form: $\bar{\Lambda} = \bar{I}_6 \Lambda$ or $\bar{\Lambda} = \Lambda' \bar{I}_6$ where $\Lambda, \Lambda' \in L(3, 3)$. Notice that the equality $[\bar{I}_6, \Lambda] = 0$ does not hold for an arbitrary $\Lambda \in L(3, 3)$. However, $\bar{I}_6 \Lambda \bar{I}_6^{-1} \in L(3, 3)$ if $\Lambda \in L(3, 3)$.

Let us return to the general case and introduce a group R_6^+ of the scaling transformations in the 6D space represented by all real matrices of the form $e^\lambda I_6$, $\lambda \in \mathbb{R}^1$. Then from the above analysis it follows that the extended Lorentz group can be represented as a semidirect product of the groups $L(3, 3)$ and $P_{ST}^+ = P_{ST} \times R_6^+$, i.e. $EL(3, 3) = L(3, 3) \rtimes P_{ST}^+$.

III. GROUP OF GENERAL LINEAR TRANSFORMATIONS OF FOUR-COMPONENT MAJORANA SPINORS

We will see below, and this is of principle for the present approach, that the covering of the extended proper orthochronous Lorentz group is a group of general linear transformations of four-component Majorana spinors. These spinors are defined by the condition (see, e.g., [21]):

$$\varphi = i\gamma^2 \varphi^* \quad (10)$$

where φ^* stands for the complex conjugate spinor. From Eq. (10) it follows that four-component Majorana spinors have only two independent complex components. The group of their general linear non-degenerated transformations [we will refer to this group as $GL(4, M)$] is represented by the 4×4 complex matrices S satisfying the following conditions:

$$\gamma^2 S^* \gamma^2 = -S, \quad \det(S) \neq 0. \quad (11)$$

Obviously, if $S \in GL(4, M)$, φ satisfies Eq. (10), and $\varphi' = S\varphi$ then φ' also satisfies Eq. (10). The following two properties of the group $GL(4, M)$ are important.

(i) $SL(2, C) \subset GL(4, M)$. Thus, the covering of the usual 4D Lorentz group is contained in $GL(4, M)$.

(ii) Let us introduce the matrix

$$U = \frac{1}{2\sqrt{2}} [(1 - \gamma^5)(1 + i\gamma^2) - i(1 + \gamma^5)\gamma^4(1 - i\gamma^2)] . \quad (12)$$

It is not difficult to check that $U^\dagger U = 1$ and $U^* \gamma^2 = -iU$ where $U^\dagger = (U^*)^T$. Using these properties we obtain that if $R = USU^\dagger$ where $S \in GL(4, M)$ then $R^* = R$ and $\det(R) \neq 0$. The opposite is also true: if $S = U^\dagger R U$, $R^* = R$, and $\det(R) \neq 0$ then $S \in GL(4, M)$. Consequently, $GL(4, M)$ is isomorphic to $GL(4, R)$. In addition, $\det(S)$ is manifestly real for all S in $GL(4, M)$.

Let us denote:

$$\bar{S} = \gamma^4 S^\dagger \gamma^4 \quad (13)$$

where $S \in GL(4, M)$. The matrices \bar{S} and \bar{S}^{-1} belong to the conjugate and conjugate-contragredient (c.-c.) representations of $GL(4, M)$, respectively. The Majorana spinors of the c.-c. representation will be denoted by χ . Notice that if $S \in SL(2, C) \subset GL(4, M)$ then $\bar{S}^{-1} = S$ but in the general case $\bar{S}^{-1} \neq S$. Thus, we have the following four types of the Majorana spinors: φ , $\bar{\varphi} = \varphi^\dagger \gamma^4$, χ , and $\bar{\chi} = \chi^\dagger \gamma^4$, which transform according to the four different representations of the group $GL(4, M)$, namely:

$$\varphi' = S \varphi, \quad \bar{\varphi}' = \bar{\varphi} \bar{S}, \quad (14)$$

$$\chi' = \bar{S}^{-1} \chi, \quad \bar{\chi}' = \bar{\chi} S^{-1}. \quad (15)$$

Consider the quantities $\bar{\varphi}_1 \alpha^K \varphi_2$ and $\bar{\chi}_1 \tilde{\alpha}_K \chi_2$ where matrices α^K and $\tilde{\alpha}_K$ are determined by Eqs. (C1) and (C2) (see Appendices A and C for the definitions and properties of these and other matrices introduced in the paper). As follows from Eqs. (14) and (15), the transformation laws of these quantities are determined by the matrix products $\bar{S} \alpha^K S$ and $S^{-1} \tilde{\alpha}_K \bar{S}^{-1}$. In Appendix D it is proved that if $S \in GL(4, M)$ then

$$\bar{S} \alpha^K S = \Lambda_L^K \alpha^L, \quad (16)$$

$$S^{-1} \tilde{\alpha}_K \bar{S}^{-1} = \tilde{\Lambda}_K^L \tilde{\alpha}_L, \quad (17)$$

where

$$\Lambda_L^K = \frac{1}{4} \text{Tr} (\bar{S} \alpha^K S \tilde{\alpha}_L) = \Lambda_L^{K*}, \quad (18)$$

$$\tilde{\Lambda}_K^L = \frac{1}{4} \text{Tr} (S^{-1} \tilde{\alpha}_K \bar{S}^{-1} \alpha^L) = \tilde{\Lambda}_K^{L*}. \quad (19)$$

In Appendix D it is also proved that the matrices Λ and $\tilde{\Lambda}$ defined by Eqs. (18) and (19) form a group, that $\tilde{\Lambda} = \Lambda^{-1}$, and that the following relations hold:

$$\Lambda_K^{K'} \eta_{K'L'} \Lambda_L^{L'} = \det(S) \eta_{KL}, \quad (20)$$

$$\tilde{\Lambda}_K^{K'} \eta_{K'L'} \tilde{\Lambda}_L^{L'} = \det(S^{-1}) \eta_{KL}. \quad (21)$$

Eqs. (20) and (21) mean that the matrices Λ and $\tilde{\Lambda}$ belong to the group $EL(3, 3)$ described in Sec. II. Thus, one can conclude that the quantities $\bar{\varphi}_1 \alpha^K \varphi_2$ and $\bar{\chi}_1 \tilde{\alpha}_K \chi_2$ transform under spinor $GL(4, M)$ transformations as a vector and a covector in the $M(3, 3)$ space, respectively.

Let us determine the relation between the groups $GL(4, M)$ and $EL(3, 3)$ more precisely. The mapping $GL(4, M) \rightarrow EL(3, 3)$ is determined by Eqs. (18) or (19). Further, the matrices $S \in GL(4, M)$ with $\det(S) = +1$ form a subgroup $SL(4, M)$ which is isomorphic to $SL(4, R)$. Since the group $SL(4, R)$ is connected and the mappings (18) and (19) are continuous, the group $SL(4, M)$ is mapped only to the one connected component of the group $EL(3, 3)$, namely *onto* the proper orthochronous Lorentz group $L_+^\uparrow(3, 3)$ (see the remark at the end of Appendix D). As a result, by including space-time permutation and scaling transformations (see Appendix D) we obtain that $GL(4, M)$ is the covering of the group $L_+^\uparrow(3, 3) \rtimes P_{ST}^+$ which is a subgroup of $EL(3, 3)$. The correspondence between the total extended Lorentz group $EL(3, 3)$ and the spinor transformations will be considered in Sec. VII.

IV. EXTENDED LORENTZ SYMMETRY AND THE PHYSICAL SPACE-TIME

Now our aim is to connect the extended Lorentz symmetry in the $M(3, 3)$ space with the physical space-time. We adopt the traditional model and regard the latter as a 4D pseudo-Riemannian manifold \mathcal{M}_4 . The vectors and tensors on this manifold and its coordinates x^μ will be labelled by the Greek letters $\mu, \nu, \dots \in \{0, 1, 2, 3\}$. In particular, $g_{\mu\nu} = g_{\mu\nu}(x)$ will be the metric with -2 signature. To introduce the spinor fields with $GL(4, M)$ symmetry on \mathcal{M}_4 , we first construct a vector bundle with \mathcal{M}_4 as a base and a 6D vector space \mathcal{F}_6 as a typical fiber (see, e.g., [22] for a review of the theory of fiber bundles and its physical applications). The fiber is defined as a direct sum: $\mathcal{F}_6 = T_x^*(\mathcal{M}_4) + \mathbb{R}^2$ where $T_x^*(\mathcal{M}_4)$ is the cotangent space of \mathcal{M}_4 at a point x and \mathbb{R}^2 is the associated plane. We will suppose that there exists a family of the preferred bases for \mathcal{F}_6 which is constructed in the following way. The set of the exact one-forms $\mathbf{e}^\mu(x) = dx^\mu$ is taken as a basis for the subspace $T_x^*(\mathcal{M}_4)$. Further, it is supposed that in \mathbb{R}^2 , a family of pairs of the linearly independent one-form fields $\mathbf{e}^u(x)$ and $\mathbf{e}^v(x)$ is given. All the pairs in this family are related by the following $SO(2)$ transformation:

$$\mathbf{e}'^u = \cos \vartheta \mathbf{e}^u - \sin \vartheta \mathbf{e}^v, \quad \mathbf{e}'^v = \sin \vartheta \mathbf{e}^u + \cos \vartheta \mathbf{e}^v \quad (22)$$

where $\vartheta = \vartheta(x)$. Thus, the family of the preferred bases for \mathcal{F}_6 is represented by the sets consisting of the six fields $\mathbf{e}^\mu(x)$, $\mathbf{e}^u(x)$, and $\mathbf{e}^v(x)$, or, in shorthand notations, by the sets $\{\mathbf{e}^F(x)\}$, $F \in \{0, 1, 2, 3, u, v\}$. It is supposed that the theory should be invariant under general coordinate transformations $x \rightarrow x'(x)$ and the local $SO(2)$ transformation (22) in \mathbb{R}^2 .

Let \mathbf{G} be the metric tensor on \mathcal{F}_6 . In terms of the preferred basis $\{\mathbf{e}^F\}$, the metric \mathbf{G} has the components $G^{FF'} = \mathbf{G}(\mathbf{e}^F, \mathbf{e}^{F'})$. It is supposed that \mathbf{G} has the signature of the metric of the $M(3, 3)$ space. So, we can always find an orthonormal basis in which the metric \mathbf{G} is proportional to $\eta^{KK'}$ [see Eq. (23) below]. Therefore, the vector space \mathcal{F}_6 can be identified with $M(3, 3)$. However, in the general case, the matrix $G^{FF'}$ cannot be taken in the diagonal or block-diagonal form because two one-forms \mathbf{e}^F and $\mathbf{e}^{F'}$ entering the preferred basis are not generally orthogonal with respect to \mathbf{G} . In particular, the components G^{0u} , G^{0v} , and G^{uv} characterize deviation of the time gradient dx^0 from the direction perpendicular to the plane \mathbb{R}^2 . The connection of $G^{FF'}$ with the metric $g_{\mu\nu}$ on the base manifold will be determined below.

An orthonormal basis $\{\mathbf{E}^K(x)\}$ for the \mathcal{F}_6 fiber at a point x of the base can be constructed with the help of a set $\{E_F^K(x)\}$ consisting of the 36 real fields which constitute the 6×6 non-singular matrix (local frame). By definition, we have: $\mathbf{E}^K = E_F^K \mathbf{e}^F$. The fields E_u^K and E_v^K are invariant under general coordinate transformations. The field E_μ^K determines a mapping of the tangent space of \mathcal{M}_4 into $M(3, 3)$. It is assumed that the local frame transforms under spinor $GL(4, M)$ gauge transformations as $E_F^K = \Lambda_L^K E_F^L$ where the matrix $\Lambda_L^K \in EL(3, 3)$ is determined by Eq. (18). The orthonormality condition for the basis $\{\mathbf{E}^K\}$ reads

$$E_F^K G^{FF'} E_{F'}^{K'} = \sigma \eta^{KK'} \quad (23)$$

where σ is a quantity transforming under $GL(4, M)$ according to $\sigma' = \det(S) \sigma$. It is introduced to ensure invariance of the metric $G^{FF'}$ under corresponding transformations of the local frame. The form of σ is specified below in Eq. (25). Inverting Eq. (23), we obtain:

$$G_{FF''} G^{F''F'} = \delta_F^{F'}, \quad G_{FF'} = \sigma^{-1} E_F^K \eta_{KK'} E_{F'}^{K'}. \quad (24)$$

The set of all frames $\{E_F^K(x)\}$ is the fiber at a point x of the frame bundle over \mathcal{M}_4 constructed from the given vector bundle. Thus, in this scheme the extended Lorentz symmetry including superluminal transformations concerns only the fiber of the frame bundle. It is convenient to combine the two real fields E_u^K and E_v^K into the one complex field $W^K = E_u^K + iE_v^K$. Then the set $\{E_\mu^K, W^K\}$ will play the role of the local frame which is the basic set of variables related to the space-time sector of the theory. Let us denote

$$\sigma = \frac{1}{2} W_K^* W^K, \quad \tau = (2\sigma)^{-1} W_K W^K \quad (25)$$

(we remind that the 6D indices are lowered and raised by the constant matrices η_{KL} and η^{KL}). According to Eq. (20), the quantity σ in Eq. (25) transforms under the $GL(4, M)$ transformations as $\sigma' = \det(S) \sigma$, while τ is invariant. Notice that Eq. (25) for σ reduces the number of the independent components of the metric $G^{FF'}$ from 21 to 20, since Eqs. (24) and (25) lead to the following constraint: $G_{uu} + G_{vv} = 2$.

Let us now define

$$e_\mu^K = E_\mu^K - \frac{1}{2} (f_\mu^* z^K + f_\mu z^{K*}), \quad (26)$$

$$z^K = \sqrt{\frac{\xi - 1}{2\xi}} (\xi W^K - h W^{K*}) \quad (27)$$

where

$$f_\mu = \sigma^{-1} z_K E_\mu^K, \quad h = \tau / \sqrt{1 - \tau^* \tau}, \quad \xi = 1 + \sqrt{1 + h^* h}. \quad (28)$$

It is easy to verify that the 6D vectors e_μ^K and z^K satisfy the following conditions:

$$z_K e_\mu^K = 0, \quad z_K z^K = 0, \quad z_K^* z^K = 2\sigma. \quad (29)$$

Eqs. (25)–(29) are valid at any values of the fields E_μ^K and W^K if $\sigma \neq 0$ and $\tau^* \tau < 1$. In this region, there is a one-to-one correspondence between the local frame $\{E_\mu^K, W^K\}$ and the set of the fields $\{e_\mu^K, z^K, f_\mu, h, \sigma\}$ constrained by the conditions (29). Owing to these constraints, the number of the independent real variables in both sets is equal to 36. Thus, the replacement of the frame $\{E_\mu^K, W^K\}$ by the set $\{e_\mu^K, z^K, f_\mu, h, \sigma\}$ used in the following is the simple change of variables. On the other hand, the set $\{e_\mu^K, z^K, f_\mu, h, \sigma\}$ is constrained in addition to Eqs. (29) only by the conditions $\sigma \neq 0$ and $\det(g_{\mu\nu}) \neq 0$ (see below). There are no additional constraints on the fields f_μ and h . This fact facilitates further analysis and explains the choice of these variables.

Formally, we have: $\mathbf{E}^K = E_F^K \mathbf{e}^F = e_F^K \mathbf{e}'^F$ where $e_u^K = \text{Re}(z^K)$ and $e_v^K = \text{Im}(z^K)$. Thus, the set $\{e_\mu^K, z^K\}$ is a new local frame which connects two bases $\{\mathbf{E}^K\}$ and $\{\mathbf{e}'^F\}$, and in

which the real and the imaginary parts of the vector z^K form the orthogonal subset. Notice, however, that the basis $\{e'^F\}$, which is transformed into $\{E^K\}$ with the help of the frame $\{e_\mu^K, z^K\}$, is not generally the preferred one. So, the fields f_μ , h , and σ are necessary to determine all the components of the metric \mathbf{G} in terms of the preferred basis.

Let us define the relation between the metric $g_{\mu\nu}$ on \mathcal{M}_4 and the local frame by the ansatz

$$g_{\mu\nu} = \sigma^{-1} e_{K\mu} e_\nu^K. \quad (30)$$

The matrix $g_{\mu\nu}$ is invertible if the matrix E_F^K is non-singular and $\sigma \neq 0$. The inverse matrix $g^{\mu\nu}$ will be used to raise the 4D coordinate indices. Using Eqs. (24)–(30), one can show after some algebra that the connection between $g_{\mu\nu}$ and the metric $G^{FF'}$ on the fiber is determined by the natural relation $G^{\mu\nu} = g^{\mu\nu}$. The other components of $G^{FF'}$ are expressed in terms of the fields h and f_μ . Invertibility of $g_{\mu\nu}$ in combination with Eqs. (29) and (30) leads to the equality

$$\sigma^{-1} e_\mu^K e_L^\mu + (2\sigma)^{-1} (z^{K*} z_L + z^K z_L^*) = \delta_L^K \quad (31)$$

which is the completeness relation for the set $\{e_\mu^K, z^K\}$. Notice that the fields h , f_μ , and $g_{\mu\nu}$ are invariant under the spinor $GL(4, M)$ transformations and contain 20 independent real components. Together with 16 parameters of the $GL(4, M)$ group, this yields 36 degrees of freedom that is equal to the number of components of the local frame $\{E_F^K\}$.

It is interesting to note that if we would define the metric on \mathcal{M}_4 as $\tilde{g}_{\mu\nu} = e_{K\mu} e_\nu^K$ we would arrive at the Weyl geometry (see, e.g., Refs. [23, 24] where the Weyl theory and its modifications are described in detail). In the present case the Weyl gauge transformation is generated by the scaling transformation of the $GL(4, M)$ group. We will not use this approach because it leads to the known difficulties. Only the metric (30), which is invariant under the $GL(4, M)$ transformations including scaling, enters the all subsequent formulas.

The set of the fields $\{e_\mu^K, z^K, f_\mu, h, \sigma\}$ constrained by Eqs. (29) can be further reduced to the following set: $\{e_\mu^a, f_\mu, h, \hat{\theta}\}$ where e_μ^a is the usual vierbein, $a \in \{1, 2, 3, 4\}$ (see Appendix A), and $\hat{\theta}$ is a set consisting of the 10 parameters of the 16-parameter gauge group $GL(4, M)$. The set $\{e_\mu^a, f_\mu, h, \hat{\theta}\}$ is determined in the following way. It is not difficult to show that if the fields $e_\mu'^K$, z'^K , and σ' satisfy Eqs. (29), they can always be represented in the form

$$e_\mu'^K = \Lambda_L^K(\hat{\theta}) e_\mu^L, \quad z'^K = \Lambda_L^K(\hat{\theta}) z^L, \quad \sigma' = \det(S(\hat{\theta})) \sigma \quad (32)$$

where

$$\sigma = 1, \quad z^a = 0, \quad z^5 = 1, \quad z^6 = i, \quad e_\mu^5 = e_\mu^6 = 0 \quad (33)$$

and the matrix $\Lambda_L^K(\hat{\theta})$ is determined by Eq. (18) with $S = S(\hat{\theta}) \in GL(4, M)$. The 10-parameter set $\hat{\theta}$ is sufficient for this representation. We will assume that the gauge (33) corresponds to zero values of the parameters $\hat{\theta}$. Thus, in the representation (32) and (33) we again have 36 independent real components including the vierbein e_μ^a , the fields f_μ and h , and the set of the gauge parameters $\hat{\theta}$.

V. SPINOR TRANSFORMATIONS AND LOCAL GAUGE SYMMETRY

Consider spinor transformations (14) and (15) in which the matrix S depends on the point of the space-time manifold. Let $D_\mu = \partial_\mu + \Omega_\mu$ be a covariant derivative of the spinor field φ where Ω_μ is the spin connection. From the condition $D'_\mu S\varphi = SD_\mu\varphi$ we obtain the following transformation law:

$$\Omega'_\mu = S\Omega_\mu S^{-1} - (\partial_\mu S)S^{-1}. \quad (34)$$

Using the complete set $\{1, S_{KL}\}$ (see Appendix C) let us represent the spin connection in the form

$$\Omega_\mu = \frac{i}{2} \Omega_\mu^{KL} S_{KL} + \lambda_\mu \quad (35)$$

where Ω_μ^{KL} and λ_μ are the gauge fields related to the matrix Ω_μ by the formulas [see Eqs. (C30) and (C32)]

$$\Omega_\mu^{KL} = -i \text{Tr}(\Omega_\mu S^{KL}), \quad \lambda_\mu = \frac{1}{4} \text{Tr}(\Omega_\mu). \quad (36)$$

Since both φ and $D_\mu\varphi$ should be Majorana spinors, the matrix Ω_μ satisfies the condition [cf. Eq. (11)] $\gamma^2 \Omega_\mu^* \gamma^2 = -\Omega_\mu$. From this it follows that the fields Ω_μ^{KL} and λ_μ are real. Further, from the definitions (18), (C5) and Eqs. (16), (17), (20), and (D4) it follows that

$$S^{-1} S^{KL} S = \det(S^{-1}) \Lambda_{K'}^K \Lambda_{L'}^L S^{K'L'} \quad (37)$$

where matrices $\Lambda_{K'}^K$ are determined by Eq. (18). Using Eqs. (18)–(20), (C5), (C9), (C31), and (D4) one can also obtain

$$\text{Tr}(S^{-1} S^{KL} \partial_\mu S) = \frac{i}{2} \det(S^{-1}) (\Lambda_{K'}^K \partial_\mu \Lambda_{L'}^L - \Lambda_{L'}^L \partial_\mu \Lambda_{K'}^K) \eta^{K'L'}. \quad (38)$$

In addition we have:

$$\text{Tr}(S^{-1} \partial_\mu S) = \det(S^{-1}) \partial_\mu \det(S). \quad (39)$$

As a result, Eqs. (34)–(39) lead to the following formulas of the transformations of the gauge fields Ω_μ^{KL} and λ_μ :

$$\Omega'_\mu^{KL} = \det(S^{-1}) [\Lambda_{K'}^K \Lambda_{L'}^L \Omega_\mu^{K'L'} - \frac{1}{2} (\Lambda_{K'}^K \partial_\mu \Lambda_{L'}^L - \Lambda_{L'}^L \partial_\mu \Lambda_{K'}^K) \eta^{K'L'}], \quad (40)$$

$$\lambda'_\mu = \lambda_\mu - \frac{1}{4} \det(S^{-1}) \partial_\mu \det(S). \quad (41)$$

We do not regard these gauge fields as independent variables here. Instead, we express them in terms of the local frame variables in analogy with the standard approach, which uses the vierbein formalism. Let us introduce the field

$$\begin{aligned} \omega_\mu^{KL} &= \frac{1}{2\sigma} (e_{\nu;\mu}^K e^{L\nu} - e_{\nu;\mu}^L e^{K\nu}) \\ &\quad - \frac{1}{4\sigma} (z^{K*} \partial_\mu z^L + z^K \partial_\mu z^{L*} - z^{L*} \partial_\mu z^K - z^L \partial_\mu z^{K*}) \end{aligned} \quad (42)$$

where $e_{\nu;\mu}^K$ is the covariant derivative with respect to the conventional metric connection $\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$, i.e., $e_{\nu;\mu}^K = \partial_\mu e_\nu^K - \Gamma_{\mu\nu}^\lambda e_\lambda^K$. We also put

$$\lambda_\mu = -\frac{1}{4}\sigma^{-1}\partial_\mu\sigma. \quad (43)$$

Taking into account that the vectors of the local frame $\{e_\mu^K, z^K\}$ and the quantity σ transform under spinor $GL(4, M)$ transformations as $e_\mu'^K = \Lambda_L^K e_\mu^L$, $z'^K = \Lambda_L^K z^L$, and $\sigma' = \det(S)\sigma$ and using Eq. (31), we obtain transformations laws (40) and (41) for the gauge fields determined by Eqs. (42) and (43).

Consider the additional $SO(2)$ symmetry determined by Eq. (22). Corresponding local $U(1)$ transformation of the fields W^K and z^K reads

$$W'^K = e^{i\vartheta}W^K, \quad z'^K = e^{i\vartheta}z^K. \quad (44)$$

Let us define (x -dependent) matrix γ_\perp as

$$\gamma_\perp = i\sigma^{-1}z^{K*}z^L S_{KL}. \quad (45)$$

Let C_μ be a gauge field and let C_μ is changed under the transformation (44) as

$$C'_\mu = C_\mu + \partial_\mu\vartheta. \quad (46)$$

Let us put

$$\Omega_\mu = \omega_\mu + \lambda_\mu + \frac{i}{2}C_\mu\gamma_\perp \quad (47)$$

where

$$\omega_\mu = \frac{i}{2}\omega_\mu^{KL}S_{KL} \quad (48)$$

and the fields ω_μ^{KL} and λ_μ are determined by Eqs. (42) and (43). From the above derivation it follows that the spin connection (47) is changed under the local $GL(4, M)$ transformations according to Eq. (34) and is invariant under the local $U(1)$ symmetry (44).

Spin connection $\tilde{\Omega}_\mu$ for the spinor field χ of the c.-c. representation is determined from the condition $\tilde{D}'_\mu\tilde{S}^{-1}\chi = \tilde{S}^{-1}\tilde{D}_\mu\chi$ [see Eq. (15)] where $\tilde{D}_\mu = \partial_\mu + \tilde{\Omega}_\mu$. Analogously to the previous case we obtain

$$\tilde{\Omega}_\mu = \tilde{\omega}_\mu - \lambda_\mu - \frac{i}{2}C_\mu\tilde{\gamma}_\perp \quad (49)$$

where

$$\tilde{\omega}_\mu = \frac{i}{2}\omega_\mu^{KL}\tilde{S}_{KL}, \quad (50)$$

$$\tilde{\gamma}_\perp = -i\sigma^{-1}z^{K*}z^L\tilde{S}_{KL}. \quad (51)$$

VI. CHIRAL SPINOR FIELDS

We now have at our disposal all necessary elements to construct the spinor action which is invariant under both local $GL(4, M)$ and general coordinate transformations. However, there are still two little problems. First, if a spinor φ satisfies constraint (10) then $i\varphi$ does not satisfy it, and consequently $i\varphi$ is not a Majorana spinor. In other words, we cannot multiply the Majorana spinors by the arbitrary complex numbers without violating the Majorana constraint. As a result, we cannot immediately introduce internal symmetries of the Majorana spinor fields based, e.g., on the unitary groups. Second, to construct a realistic theory one needs chiral spinor fields. In the standard approach their role is played by the Weyl spinors. The four-component Weyl spinors have the same number of the independent complex components, namely two, as the Majorana spinors, but they obey different constraint which is not invariant under the $GL(4, M)$ transformations (see, e.g., [25] for the analysis of the questions related to the chirality and the properties of the different types of the spinors).

These problems can be solved in the following way. First of all, notice that from the definitions (45) and (51) and Eqs. (29) and (C13) it follows that

$$\gamma_{\perp}^2 = \tilde{\gamma}_{\perp}^2 = 1. \quad (52)$$

Therefore, the matrices

$$\Pi_{\pm} = \frac{1}{2} (1 \pm \gamma_{\perp}), \quad \tilde{\Pi}_{\pm} = \frac{1}{2} (1 \pm \tilde{\gamma}_{\perp}) \quad (53)$$

are the projection operators. In the gauge (33), using definitions of Appendix C we obtain

$$\gamma_{\perp} = \tilde{\gamma}_{\perp} = \gamma^5, \quad (54)$$

$$\Pi_{-} = \tilde{\Pi}_{-} = P_L, \quad \Pi_{+} = \tilde{\Pi}_{+} = P_R \quad (55)$$

where P_L and P_R are the usual chiral projection operators.

Let us introduce generalized Weyl spinors φ_{\pm} and the spinors of the c.-c. representation χ_{\pm} as follows:

$$\varphi_{\pm} = \Pi_{\pm} \varphi, \quad \chi_{\pm} = \tilde{\Pi}_{\pm} \chi \quad (56)$$

where φ and χ are the Majorana spinors. Obviously, they obey the following generalized Weyl constraint:

$$\gamma_{\perp} \varphi_{\pm} = \pm \varphi_{\pm}, \quad \tilde{\gamma}_{\perp} \chi_{\pm} = \pm \chi_{\pm}. \quad (57)$$

In the gauge (33) φ_{-} and χ_{-} will be the left-handed spinors and φ_{+} and χ_{+} will be the right-handed ones. So, in what follows we will also use notations: $\varphi_L = \varphi_{-}$ and $\chi_R = \chi_{+}$. Thus, the spinors φ_{\pm} and χ_{\pm} play the role of the chiral spinor fields in the approach under consideration. Notice that in order to introduce the chiral spinors within this approach it is of principle to reduce the dimensionality of the physical space-time to four as compared with the 6D fiber.

The generalized Weyl spinors transform under the $GL(4, M)$ transformations like the Majorana spinors [see Eqs. (14) and (15)], i.e.

$$\varphi'_\pm = \Pi'_\pm \varphi' = S \varphi_\pm, \quad (58)$$

$$\chi'_\pm = \tilde{\Pi}'_\pm \chi' = \bar{S}^{-1} \chi_\pm \quad (59)$$

where the matrices Π'_\pm and $\tilde{\Pi}'_\pm$ are defined by Eqs. (45), (51), and (53) in which σ and z^K are replaced by $\sigma' = \det(S) \sigma$ and $z'^K = \Lambda^K_L z^L$. This replacement yields

$$\gamma'_\pm = S \gamma_\pm S^{-1}, \quad \tilde{\gamma}'_\pm = \bar{S}^{-1} \tilde{\gamma}_\pm \bar{S}. \quad (60)$$

From Eqs. (58)–(60) it follows that the constraint (57) is invariant under the $GL(4, M)$ transformations.

Despite the fact that the spinors φ_\pm and χ_\pm transform like the Majorana spinors, they do not satisfy the Majorana constraint (10). Instead, we have

$$\varphi_\pm = i\gamma^2 \varphi_\mp^*, \quad \chi_\pm = i\gamma^2 \chi_\mp^*. \quad (61)$$

An important property of the generalized Weyl spinors is that in contrast to the Majorana spinors they admit multiplication by the complex numbers. Indeed, from Eqs. (52), (53), and (56) it follows that

$$\varphi'_\pm = e^{\pm i\theta} \varphi_\pm = \Pi_\pm \varphi' \quad (62)$$

where $\varphi' = S(\theta)\varphi$, $S(\theta) = \exp(i\theta \gamma_\pm)$, and θ is real. Using the definition (45) one can check that $S(\theta) \in GL(4, M)$. Then both φ and φ' are the Majorana spinors, and consequently both φ_\pm and φ'_\pm in Eq. (62) are the generalized Weyl spinors. The same is true for the spinors χ_\pm . Taking into account this property, we will assume that the spinors $\varphi_L = \varphi_-$ and $\chi_R = \chi_+$ transform under the phase transformation (44) as follows

$$\varphi'_L = e^{i(q-\frac{1}{2})\vartheta} \varphi_L, \quad \chi'_R = e^{i(q-\frac{1}{2})\vartheta} \chi_R \quad (63)$$

where q is a reduced gauge charge.

VII. CONSTRUCTION OF THE SPINOR ACTION AND THE DISCRETE SYMMETRIES

Let us define the following x -dependent matrices:

$$\beta_\mu = \frac{i}{2} \sigma^{-2} e_\mu^K z^L z^{M*} \beta_{KLM}, \quad \tilde{\beta}_\mu = \frac{i}{2} \sigma^{-1} e_\mu^K z^L z^{M*} \tilde{\beta}_{KLM}. \quad (64)$$

These matrices are invariant under the simultaneous $GL(4, M)$ transformations of the type (16) and (17) and the $EL(3, 3)$ transformations of the local frame. In the gauge (33) they take the form

$$\beta_\mu = \tilde{\beta}_\mu = \gamma_\mu, \quad \gamma_\mu = e_\mu^a \gamma_a. \quad (65)$$

Consider the spinor Lagrangian of the following form (in units where $\hbar = 1$ and $c = 1$):

$$\mathcal{L}_{\text{spin}} = \frac{i}{2} \left(\bar{\varphi}_L \beta^\mu \nabla_\mu \varphi_L + \bar{\chi}_R \tilde{\beta}^\mu \tilde{\nabla}_\mu \chi_R \right) - m \bar{\chi}_R \varphi_L + \text{H.c.} \quad (66)$$

where m is a mass parameter,

$$\nabla_\mu = \partial_\mu + \omega_\mu - i q C_\mu, \quad \tilde{\nabla}_\mu = \partial_\mu + \tilde{\omega}_\mu - i q C_\mu, \quad (67)$$

and the spin connections ω_μ and $\tilde{\omega}_\mu$ are defined by Eqs. (48) and (50). The spinor fields φ_L and χ_R are supposed to be anticommuting. The covariant derivatives ∇_μ and $\tilde{\nabla}_\mu$ differ from the derivatives D_μ and \tilde{D}_μ introduced in Sec. V first, by the absence of the gauge field λ_μ disappearing due to the hermiticity of the Lagrangian $\mathcal{L}_{\text{spin}}$. Second, in Eq. (67) the properties (57) of the generalized Weyl spinors were taken into account and the charges under the $U(1)$ gauge field C_μ were changed in order to compensate additional contributions arising from the phase transformations (63).

Corresponding spinor action reads

$$\mathcal{S}_{\text{spin}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{spin}} \quad (68)$$

where $g = \det(g_{\mu\nu})$. By construction, this action is invariant under both general 4D coordinate and local $GL(4, M)$ spinor transformations. In addition it is invariant under the local $U(1)$ transformations defined by Eqs. (44) and (63). On the other hand, from Eq. (65) it follows that in the gauge (33) and at $q = 0$, $\mathcal{S}_{\text{spin}}$ reduces to the action of the standard 4D theory for the massive neutral fermions without additional internal symmetries.

Consider the discrete symmetries of the Lagrangian (66). We will consider only the symmetries concerning transformations of the spinors and the vectors in the fiber of the bundle, since the invariance of the theory with respect to the general coordinate transformations is implied. Let us define the following simultaneous transformation of the Majorana spinors φ_L and χ_R and the vectors of the local frame $\{E_\mu^K, W^K\}$:

$$\varphi'_L = i\gamma^4 \chi_R, \quad \chi'_R = i\gamma^4 \varphi_L, \quad (69)$$

$$E'^K_\mu = \sigma^{-1} \mathcal{P}^K_L E^L_\mu, \quad W'^K = \sigma^{-1} \mathcal{P}^K_L W^L \quad (70)$$

where the matrix of the space reflection \mathcal{P} and the quantity σ are defined by Eqs. (8) and (25), respectively. The transformation (70) is a combination of the space reflection and the non-linear conformal transformation $E'^K_\mu = \sigma^{-1} E^K_\mu$, $W'^K = \sigma^{-1} W^K$. So, in what follows it will be referred to as P_C transformation. The combination of the transformations (69) and (70) will be referred to as P transformation. Using definitions (42), (48), (50), and (64) and the properties (C28) and (C29) one can show that the matrices β_μ , $\tilde{\beta}_\mu$, ω_μ , and $\tilde{\omega}_\mu$ are changed under the P_C transformation as follows:

$$\beta'_\mu = \gamma^4 \tilde{\beta}_\mu \gamma^4, \quad \tilde{\beta}'_\mu = \gamma^4 \beta_\mu \gamma^4, \quad (71)$$

$$\omega'_\mu = \gamma^4 \tilde{\omega}_\mu \gamma^4, \quad \tilde{\omega}'_\mu = \gamma^4 \omega_\mu \gamma^4. \quad (72)$$

Since the fields φ_L and χ_R enter the Lagrangian (66) in a symmetric way, from Eqs. (71) and (72) it follows that $\mathcal{L}_{\text{spin}}$ is invariant under the P transformation. This invariance can be broken if, for instance, the fields φ_L and χ_R possess different internal symmetries.

The PT transformation is defined as a combination of the spinor transformation

$$\varphi'_L = i\gamma^2 \varphi_L^*, \quad \chi'_R = -i\gamma^2 \chi_R^*, \quad (73)$$

the permutation of the spinor fields in each term of $\mathcal{L}_{\text{spin}}$, the space-time reflection in the fiber

$$E'^K_\mu = -E^K_\mu, \quad W'^K = -W^K, \quad (74)$$

and the change of the sign of the field C_μ . From Eqs. (C17), (C19), (C20), (C25), and (C26) it follows that the spinor Lagrangian is invariant under the PT transformation defined in this way.

The CPT transformation is a combination of the spinor transformation

$$\varphi'_L = \varphi_L, \quad \chi'_R = -\chi_R, \quad (75)$$

the permutation of the spinor fields in $\mathcal{L}_{\text{spin}}$, and the space-time reflection (74). Obviously, the Lagrangian (66) is invariant under this transformation.

Finally, notice that, as shown in Sec. II, the group $L(3,3)$ can be constructed from the proper orthochronous Lorentz group $L_+^\uparrow(3,3)$ by adding the space and the space-time reflections. Thus, we arrive at the result: the group $GL(4, M)$ supplemented by the P and the PT transformations is the covering of the total extended Lorentz group $EL(3,3)$.

VIII. EXTRA FIELDS AS THE DARK MATTER CANDIDATES

Consider now the actions for the other fields introduced in Sections IV and V. Let us take them in the simplest form. In particular, for the gravitational action we adopt the standard Einstein-Hilbert ansatz

$$\mathcal{S}_g = -\frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R \quad (76)$$

where R is the Ricci scalar curvature of the space-time with metric $g_{\mu\nu}$:

$$R = g^{\mu\nu} (\partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\lambda\kappa}^\kappa \Gamma_{\mu\nu}^\lambda - \Gamma_{\mu\kappa}^\lambda \Gamma_{\lambda\nu}^\kappa) \quad (77)$$

and G_N is the Newtonian constant. There remain three extra fields: h , f_μ , and C_μ . In contrast to the Kaluza-Klein theories (see, e.g., [26]), these fields are not connected with the metric $g_{\mu\nu}$ by any symmetry conditions. So, the Lagrangian for them can be constructed independently.

Let us define the $U(1)$ covariant derivative of the field h :

$$h_{|\mu} = (\partial_\mu - 2iC_\mu) h \quad (78)$$

and the strength tensors of the fields f_μ and C_μ :

$$f_{[\mu|\nu]} = (\partial_\nu - iC_\nu) f_\mu - (\partial_\mu - iC_\mu) f_\nu, \quad (79)$$

$$C_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu. \quad (80)$$

The simplest form of the action including extra fields is:

$$\mathcal{S}_{\text{ext}} = \int d^4x \sqrt{-g} \mathcal{L}_{\text{ext}} \quad (81)$$

where

$$\mathcal{L}_{\text{ext}} = \frac{1}{16\pi G_N} [\alpha_h (h_{|\mu}^* h^{|\mu} - m_h^2 h^* h) - \alpha_f (\frac{1}{2} f_{[\mu|\nu]}^* f^{[\mu|\nu]} - m_f^2 f_\mu^* f^\mu)] - \frac{1}{16\pi\alpha_C} C_{\mu\nu} C^{\mu\nu}. \quad (82)$$

In Eq. (82), α_h , α_f , and α_C are the dimensionless constants and m_h and m_f are the mass parameters. \mathcal{L}_{ext} is the standard Lagrangian describing two massive complex bosonic fields (scalar and vector) and one massless field of the Maxwell type. However, there are no apparent reasons to identify the field C_μ with the usual electromagnetic field or with the $U(1)_Y$ gauge field of the Standard Model [27]. It is more natural to associate the extra fields h , f_μ , and C_μ with the fields of the invisible (dark) matter. Notice that these fields do not correspond to tachyons in spite of the superluminal Lorentz symmetry of the theory.

Consider the interaction of the extra fields with the ordinary (visible) matter represented here by the fermionic fields discussed above. This interaction is mediated by the local frame $\{E_\mu^K, W^K\}$ and the $U(1)$ gauge field C_μ . The fields h and f_μ are expressed in terms of the local frame via Eqs. (25), (27), and (28). On the other hand, the fermionic fields are connected with the local frame by means of Eqs. (25)–(28), (30), (42), (45), (48), (50), (51), (53), (56), (64), (66), and (67). But in the gauge (33) this connection reduces to the dependence of $\mathcal{L}_{\text{spin}}$ on the vierbein e_μ^a , while the fields h and f_μ become independent variables. Thus, in this gauge the fermions are connected with the fields h and f_μ only via the metric $g_{\mu\nu}$ (i.e., via gravity) and via the field C_μ . The $U(1)$ charges of the fields h and f_μ are equal to 2 and 1, respectively, (in units of $\sqrt{\alpha_C}$) while the charge of the fermionic field is equal to q . In principle, the value of q is arbitrary and it can be different for the different kinds of the fermions. If the value of $q\sqrt{\alpha_C}$ is small or equal to zero, the interaction between the extra fields and the ordinary matter is weak. It allows us to consider the extra fields as the dark matter candidates.

There are variety of models predicting the existence of the non-baryonic DM particles. Among those one should mention the supersymmetric models and the models with universal extra dimensions of the Kaluza-Klein type (see [18, 26, 28] for a review), axion models [19, 29], and the model of the mirror matter [30, 31]. Perhaps there exist several kinds of the fields corresponding to the DM particles. Notice that the possibility of the application of the 6D relativity [6, 7, 8, 9] in the $M(3, 3)$ space to the DM problem was discussed in Ref. [32]. In the present paper the same $M(3, 3)$ space is introduced. However, in contrast to Refs. [6, 7, 8, 9, 10, 11, 12], this space is related here to the fiber of the bundle but not to the

physical space-time. In particular for this reason, the present treatment of the consequences of the theory in terms of the DM hypothesis differs from the approach of Ref. [32].

Let us emphasize that the Lagrangians of the fermionic, gravitational, and extra fields were taken here in the simplest form. In particular, the internal symmetries of the fermionic fields were not taken into account and the possible couplings of the extra $U(1)$ gauge field C_μ to the gauge fields of these internal symmetries and to the Higgs bosons were not considered. The couplings of this and other types were studied in the extensions of the Standard Model which include an extra $U(1)$ gauge boson (see [33, 34, 35, 36] and references therein). If these couplings are introduced, the field C_μ can acquire a mass due to the spontaneous breaking of the $U(1)$ symmetry. In addition, in principle, it is possible to introduce direct couplings of the extra fields h and f_μ to each other and their Yukawa couplings to the fermionic fields (of the ordinary or the dark matter) which have different charges q under the extra $U(1)$ symmetry (see, e.g., discussion of the similar couplings in Ref. [37] in the context of the model of the scalar DM). However, the analysis of these questions is beyond the scope of the present work.

IX. SUMMARY

A central question of the paper is how to include the extended superluminal Lorentz symmetry in the four-dimensional (4D) theory. It is known that this symmetry can be consistently introduced in the 6D Minkowski space $M(3,3)$. In this paper it is shown that the problem of reconciling the 6D relativity with the 4D theory can be solved if it is assumed that the extended Lorentz group acts only in the fiber of the bundle for which the physical space-time serves as a base. The latter is regarded as a 4D pseudo-Riemannian manifold. In this approach, a source of the extended Lorentz symmetry is a group of general linear transformations of four-component Majorana spinors. The fiber bundle allows us to construct the action including the spinor fields with the extended symmetry on the 4D manifold. It is shown that the spinor part of the action reduces to the spinor action of the standard 4D theory under a certain choice of the gauge.

The action obtained contains extra fields describing in the general case two massive particles (4D vector and scalar ones) weakly interacting with ordinary particles. The third extra $U(1)$ gauge field is massless. There are no tachyons in the theory. If the charges of the fermionic fields under the extra $U(1)$ symmetry are equal to zero, the extra fields interact with the ordinary matter only via gravity. It means that they can be considered as the dark matter candidates.

Acknowledgments

The author is grateful to Prof. V. S. Barashnikov for references on six-dimensional relativity in the first stage of this work.

APPENDIX A: NOTATIONS AND CONVENTIONS

The following notations are used throughout the paper. Capital Latin letters K, L, M, N represent indices of the flat $M(3, 3)$ space ($K \in \{1, 2, 3, 4, 5, 6\}$) with metric

$$\eta_{KL} = \text{diag}\{-1, -1, -1, +1, +1, +1\}. \quad (\text{A1})$$

All the 6D indices are lowered and raised by the matrix η_{KL} and its inverse $\eta^{KL} = \eta_{KL}$. Small Latin letters a, b, c, d indicate indices of the flat 4D space ($a \in \{1, 2, 3, 4\}$) with metric

$$\eta'_{ab} = \text{diag}\{-1, -1, -1, +1\}. \quad (\text{A2})$$

The following representation for the 4×4 Dirac matrices is used:

$$\gamma^k = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\sigma}^k \\ \boldsymbol{\sigma}^k & \mathbf{0} \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad (\text{A3})$$

where $k = 1, 2, 3$, and boldface characters denote 2×2 matrices including conventional Pauli matrices $\boldsymbol{\sigma}^k$:

$$\boldsymbol{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A4})$$

APPENDIX B: PROPERTIES OF THE DETERMINANTS OF THE $O(p, q)$ MATRICES

Let C be a real matrix of the matrix representation of the group $O(p, q)$, i.e. the following equality is fulfilled:

$$C^T \eta C = \eta \quad (\text{B1})$$

where

$$\eta = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad (\text{B2})$$

I_p and I_q are the $p \times p$ and $q \times q$ identity matrices. Let us write C in the block form as

$$C = \begin{pmatrix} A & b \\ a & B \end{pmatrix} \quad (\text{B3})$$

where A , b , a , and B are the $p \times p$, $p \times q$, $q \times p$, and $q \times q$ matrices, respectively. Eqs. (B1)–(B3) lead to the equalities:

$$A^T A = I_p + a^T a, \quad (\text{B4})$$

$$B^T B = I_q + b^T b, \quad (\text{B5})$$

$$a^T B = A^T b. \quad (\text{B6})$$

The matrix $a^T a$ is real, symmetric, and non-negative-definite. Consequently, according to Eq. (B4), all the eigenvalues of the matrix $A^T A$ are equal to or greater than unity. From this we obtain that $|\det(A)| \geq 1$. Analogously one can prove that $|\det(B)| \geq 1$.

The following formula for the determinant of the matrix of the form (B3) is known (see, e.g., [20]):

$$\det(C) = \det(A) \det(B - aA^{-1}b). \quad (\text{B7})$$

Using Eq. (B6) we get $B^T a A^{-1} b = b^T b$. From this and from Eq. (B5) it follows that $B^T (B - aA^{-1}b) = B^T B - b^T b = I_q$. Therefore $\det(B - aA^{-1}b) = 1/\det(B)$ and from Eq. (B7) we find the final result: $\det(C) = \det(A)/\det(B)$. Analogously one can obtain that $\det(C) = \det(B)/\det(A)$. Because $|\det(C)| = 1$ we also get that $|\det(A)| = |\det(B)|$.

APPENDIX C: MATRICES OF THE SPINOR TRANSFORMATIONS AND THEIR BASIC PROPERTIES

Let us introduce 4×4 matrices α^K and $\tilde{\alpha}^K$

$$\alpha^A = \gamma^5 \gamma^A, \quad \alpha^6 = i \gamma^5, \quad (\text{C1})$$

$$\tilde{\alpha}^A = \gamma^A \gamma^5, \quad \tilde{\alpha}^6 = -i \gamma^5 \quad (\text{C2})$$

where $A \in \{1, 2, 3, 4, 5\}$ and the representation of the Dirac matrices given in Appendix A is implied. Using the above formulas, let us define the following matrices:

$$\beta^{KLM} = \frac{i}{2} \left(\delta_{K'}^K \delta_{M'}^M - \delta_{M'}^K \delta_{K'}^M \right) \alpha^{K'} \tilde{\alpha}^L \alpha^{M'}, \quad (\text{C3})$$

$$\tilde{\beta}^{KLM} = \frac{i}{2} \left(\delta_{K'}^K \delta_{M'}^M - \delta_{M'}^K \delta_{K'}^M \right) \tilde{\alpha}^{K'} \alpha^L \tilde{\alpha}^{M'}, \quad (\text{C4})$$

$$S^{KL} = \frac{i}{2} \left(\eta^{KL} - \tilde{\alpha}^L \alpha^K \right), \quad (\text{C5})$$

$$\tilde{S}^{KL} = \frac{i}{2} \left(\alpha^K \tilde{\alpha}^L - \eta^{KL} \right). \quad (\text{C6})$$

From these definitions and the properties of the Dirac matrices it follows that:

- (a) Matrices β^{KLM} , $\tilde{\beta}^{KLM}$, S^{KL} , and \tilde{S}^{KL} are totally antisymmetric in the 6D indices K, L, M . The matrices β^{KLM} ($\tilde{\beta}^{KLM}$) have properties of the anti-self-dual (self-dual) tensors in the 6D indices, i.e.:

$$\beta^{KLM} = -\frac{1}{6} \varepsilon^{KLMK'L'M'} \beta_{K'L'M'}, \quad (\text{C7})$$

$$\tilde{\beta}^{KLM} = \frac{1}{6} \varepsilon^{KLMK'L'M'} \tilde{\beta}_{K'L'M'} \quad (\text{C8})$$

where $\varepsilon^{KLMK'L'M'}$ is the totally antisymmetric 6D tensor with $\varepsilon^{123456} = -1$. There are only 10 independent matrices β^{KLM} and 15 independent matrices S^{KL} . The same is true for the matrices $\tilde{\beta}^{KLM}$ and \tilde{S}^{KL} . In the explicit form we have:

$$\alpha^a = -\tilde{\alpha}^a = \gamma^5 \gamma^a, \quad \alpha^5 = \tilde{\alpha}^5 = 1, \quad \alpha^6 = -\tilde{\alpha}^6 = i \gamma^5,$$

$$\begin{aligned}\beta^{a56} &= \tilde{\beta}^{a56} = \gamma^a, & \beta^{ab5} &= \tilde{\beta}^{ab5} = i\sigma^{ab}, \\ \beta^{ab6} &= -\tilde{\beta}^{ab6} = \frac{i}{2}\varepsilon^{abcd}\sigma_{cd}, & \beta^{abc} &= -\tilde{\beta}^{abc} = -\varepsilon^{abcd}\gamma_d,\end{aligned}$$

$$\begin{aligned}S^{ab} &= \tilde{S}^{ab} = \frac{i}{2}\sigma^{ab}, & S^{a5} &= -\tilde{S}^{a5} = -\frac{i}{2}\gamma^5\gamma^a, \\ S^{a6} &= \tilde{S}^{a6} = -\frac{1}{2}\gamma^a, & S^{56} &= -\tilde{S}^{56} = -\frac{1}{2}\gamma^5\end{aligned}$$

where $\sigma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$ and ε^{abcd} is the totally antisymmetric 4D tensor with $\varepsilon^{1234} = -1$.

- (b) Matrices $\{\alpha^K, \beta^{KLM}\}$, $\{\tilde{\alpha}^K, \tilde{\beta}^{KLM}\}$, $\{1, S^{KL}\}$, and $\{1, \tilde{S}^{KL}\}$ form four complete sets. This means that for any 4×4 complex matrices A and B the following equalities are fulfilled:

$$\text{Tr}(A\alpha^K) \text{Tr}(B\tilde{\alpha}_K) + \frac{1}{12} \text{Tr}(A\beta^{KLM}) \text{Tr}(B\tilde{\beta}_{KLM}) = 4 \text{Tr}(AB), \quad (\text{C9})$$

$$\text{Tr}(A) \text{Tr}(B) + 2 \text{Tr}(AS^{KL}) \text{Tr}(BS_{KL}) = 4 \text{Tr}(AB), \quad (\text{C10})$$

$$\text{Tr}(A) \text{Tr}(B) + 2 \text{Tr}(A\tilde{S}^{KL}) \text{Tr}(B\tilde{S}_{KL}) = 4 \text{Tr}(AB). \quad (\text{C11})$$

- (c) Matrices S^{KL} and \tilde{S}^{KL} are the generators of the Lie algebra $so(3,3)$, i.e., we have:

$$[S^{KL}, S^{K'L'}] = i(\eta^{KL'} S^{LK'} + \eta^{LK'} S^{KL'} - \eta^{KK'} S^{LL'} - \eta^{LL'} S^{KK'}) \quad (\text{C12})$$

and the same for the matrices \tilde{S}^{KL} .

The last equation follows from the general formula for the product of the S^{KL} matrices

$$\begin{aligned}S^{KL} S^{K'L'} &= \frac{1}{4} (\eta^{KK'} \eta^{LL'} - \eta^{KL'} \eta^{LK'}) \\ &+ \frac{i}{2} (\eta^{KL'} S^{LK'} + \eta^{LK'} S^{KL'} - \eta^{KK'} S^{LL'} - \eta^{LL'} S^{KK'}) \\ &+ \frac{i}{4} \varepsilon^{KLL'LMN} S_{MN}\end{aligned} \quad (\text{C13})$$

which can be obtained using the algebra of the Dirac matrices. The formula for the product of the \tilde{S}^{KL} matrices has the same form but with the opposite sign in front of the last term. Two more important relations also follow from the algebra of the Dirac matrices, namely

$$\alpha^K S^{LM} - \tilde{S}^{LM} \alpha^K = i(\eta^{KL} \alpha^M - \eta^{KM} \alpha^L), \quad (\text{C14})$$

$$\tilde{\alpha}^K \tilde{S}^{LM} - S^{LM} \tilde{\alpha}^K = i(\eta^{KL} \tilde{\alpha}^M - \eta^{KM} \tilde{\alpha}^L). \quad (\text{C15})$$

Consider the symmetry properties of the α , β , and S matrices. Let us define for an arbitrary 4×4 matrix A the matrix $A^{(T)}$ as

$$A^{(T)} = -\gamma^4 \gamma^2 A^T \gamma^2 \gamma^4. \quad (\text{C16})$$

It is easy to check that the following relations are fulfilled:

$$(A^{\langle T \rangle})^{\langle T \rangle} = A, \quad (AB)^{\langle T \rangle} = B^{\langle T \rangle} A^{\langle T \rangle}, \quad (\text{C17})$$

$$(\alpha^K)^{\langle T \rangle} = \alpha^K, \quad (\tilde{\alpha}^K)^{\langle T \rangle} = \tilde{\alpha}^K, \quad (\text{C18})$$

$$(\beta^{KLM})^{\langle T \rangle} = -\beta^{KLM}, \quad (\tilde{\beta}^{KLM})^{\langle T \rangle} = -\tilde{\beta}^{KLM}, \quad (\text{C19})$$

$$(S^{KL})^{\langle T \rangle} = -\tilde{S}^{KL}, \quad (\tilde{S}^{KL})^{\langle T \rangle} = -S^{KL}. \quad (\text{C20})$$

Complex conjugation leads to the equalities:

$$\gamma^2 (\alpha^K)^* \gamma^2 = -\alpha^K, \quad \gamma^2 (\tilde{\alpha}^K)^* \gamma^2 = -\tilde{\alpha}^K, \quad (\text{C21})$$

$$\gamma^2 (\beta^{KLM})^* \gamma^2 = \beta^{KLM}, \quad \gamma^2 (\tilde{\beta}^{KLM})^* \gamma^2 = \tilde{\beta}^{KLM}, \quad (\text{C22})$$

$$\gamma^2 (S^{KL})^* \gamma^2 = S^{KL}, \quad \gamma^2 (\tilde{S}^{KL})^* \gamma^2 = \tilde{S}^{KL}. \quad (\text{C23})$$

From Eqs. (C16), (C18)–(C23) it follows that

$$\gamma^4 (\alpha^K)^\dagger \gamma^4 = \alpha^K, \quad \gamma^4 (\tilde{\alpha}^K)^\dagger \gamma^4 = \tilde{\alpha}^K, \quad (\text{C24})$$

$$\gamma^4 (\beta^{KLM})^\dagger \gamma^4 = \beta^{KLM}, \quad \gamma^4 (\tilde{\beta}^{KLM})^\dagger \gamma^4 = \tilde{\beta}^{KLM}, \quad (\text{C25})$$

$$\gamma^4 (S^{KL})^\dagger \gamma^4 = \tilde{S}^{KL}, \quad \gamma^4 (\tilde{S}^{KL})^\dagger \gamma^4 = S^{KL}. \quad (\text{C26})$$

On the other hand, using Eqs. (C1)–(C6) one can show that the following equalities are fulfilled:

$$\gamma^4 \alpha^K \gamma^4 = \mathcal{P}_{K'}^K \tilde{\alpha}^{K'}, \quad \gamma^4 \tilde{\alpha}^K \gamma^4 = \mathcal{P}_{K'}^K \alpha^{K'}, \quad (\text{C27})$$

$$\gamma^4 \beta^{KLM} \gamma^4 = \mathcal{P}_{K'}^K \mathcal{P}_{L'}^L \mathcal{P}_{M'}^M \tilde{\beta}^{K'L'M'}, \quad \gamma^4 \tilde{\beta}^{KLM} \gamma^4 = \mathcal{P}_{K'}^K \mathcal{P}_{L'}^L \mathcal{P}_{M'}^M \beta^{K'L'M'}, \quad (\text{C28})$$

$$\gamma^4 S^{KL} \gamma^4 = \mathcal{P}_{K'}^K \mathcal{P}_{L'}^L \tilde{S}^{K'L'}, \quad \gamma^4 \tilde{S}^{KL} \gamma^4 = \mathcal{P}_{K'}^K \mathcal{P}_{L'}^L S^{K'L'} \quad (\text{C29})$$

where the matrix of the space reflection \mathcal{P} is defined by Eq. (8).

Finally, for the traces we have

$$\text{Tr}(S^{KL}) = \text{Tr}(\tilde{S}^{KL}) = 0, \quad (\text{C30})$$

$$\text{Tr}(\beta^{KLM} \tilde{\alpha}^N) = \text{Tr}(\tilde{\beta}^{KLM} \alpha^N) = 0, \quad (\text{C31})$$

$$\text{Tr}(S^{KL} S^{MN}) = \text{Tr}(\tilde{S}^{KL} \tilde{S}^{MN}) = \eta^{KM} \eta^{LN} - \eta^{KN} \eta^{LM}, \quad (\text{C32})$$

$$\text{Tr}(\alpha^K \tilde{\alpha}^L) = 4 \eta^{KL}. \quad (\text{C33})$$

APPENDIX D: PROOF OF THE EQUATIONS CONNECTING $GL(4, M)$ AND $EL(3, 3)$ GROUPS

Let us prove Eqs. (16) and (18). First of all, notice that in accordance with Eqs. (11), (13), and (C16) we have $S^{(T)} = \bar{S}$ if $S \in GL(4, M)$. Therefore [see Eqs. (C17) and (C18)]:

$$(\bar{S} \alpha^K S)^{(T)} = \bar{S} \alpha^K S. \quad (D1)$$

Expanding the matrix $\bar{S} \alpha^K S$ in the complete set $\{\alpha^K, \beta^{KLM}\}$, we obtain Eq. (16) because the terms containing β^{KLM} vanish due to the opposite signs in the symmetry relations (C18), (C19), and (D1). Expression (18) for the matrix Λ follows from Eqs. (16) and (C33). Finally, reality of Λ follows from Eqs. (11) and (C21). In the same way we obtain Eqs. (17) and (19).

Consider the properties of the matrices Λ and $\tilde{\Lambda}$ defined by Eqs. (18) and (19). First of all, the matrices Λ form a group. To see this, let us take the matrix Λ in form (18) and the matrix Λ' in the form:

$$\Lambda'^K_L = \frac{1}{4} \text{Tr} (\bar{S}' \alpha^K S' \tilde{\alpha}_L). \quad (D2)$$

Using the completeness relation (C9), Eqs. (16), (18), (C31), and (D2) we obtain:

$$\Lambda^K_L \Lambda'^L_M = \Lambda''^K_M = \frac{1}{4} \text{Tr} (\bar{S}'' \alpha^K S'' \tilde{\alpha}_M) \quad (D3)$$

where $S'' = SS'$. Analogously, one can show that the matrices $\tilde{\Lambda}$ also form a group. In the same way, but with the use of Eqs. (17), (19), and (C33) we get

$$\Lambda^K_L \tilde{\Lambda}^L_M = \tilde{\Lambda}^K_L \Lambda^L_M = \delta^K_M. \quad (D4)$$

Thus, $\tilde{\Lambda} = \Lambda^{-1}$.

Let us prove Eq. (20). At first, consider the case $\det(S) = +1$. It is known (see, e.g., [20]) that any real matrix R with $\det(R) = 1$ can be represented in the form: $R = O_1 D O_2$ where the matrices O_1 , D , and O_2 are real, O_1 and O_2 are orthogonal, D is diagonal and positive-definite, and $\det(O_1) = \det(O_2) = \det(D) = 1$. Any real orthogonal matrix O with $\det(O) = 1$ can be represented in the form: $O = e^A$ where $A = A^* = -A^T$. From this it follows that R can be represented in the form: $R = e^A e^{A'} e^{A''}$ where the matrices A , A' , and A'' are real and traceless. Using this result and the isomorphism between $GL(4, M)$ and $GL(4, R)$, we obtain that any matrix $S \in SL(4, M)$ [i.e., any $S \in GL(4, M)$ with $\det(S) = 1$] can be represented in the form

$$S = e^\Theta e^{\Theta'} e^{\Theta''} \quad (D5)$$

where

$$\gamma^2 \Theta^* \gamma^2 = -\Theta, \quad \text{Tr}(\Theta) = 0, \quad (D6)$$

and the same conditions are fulfilled for Θ' and Θ'' . Thus, by virtue of the group properties, it is sufficient to prove Eq. (20) for $S = e^\Theta$.

From the completeness of the set $\{1, S^{KL}\}$ and from Eqs. (C23) and (C30) it follows that any 4×4 complex matrix Θ satisfying conditions (D6) can be represented in the form

$$\Theta = i \theta_K^L S_L^K \quad (\text{D7})$$

where $S_L^K = S^{KL'} \eta_{L'L}$ and the matrix θ_K^L is real and possesses the property

$$\theta_K^L = -\eta^{LL'} \eta_{KK'} \theta_{L'}^{K'}. \quad (\text{D8})$$

In shorthand notation we have:

$$\eta \theta \eta = -\theta^T. \quad (\text{D9})$$

Using Eqs. (C14), (D7), and (D8) we obtain the following equality:

$$\alpha^K e^\Theta = e^{-\bar{\Theta}} \Lambda_L^K(\theta) \alpha^L \quad (\text{D10})$$

where $\bar{\Theta} = \gamma^4 \Theta^\dagger \gamma^4$, $\Lambda(\theta) = e^{2\theta}$. Eq. (D10) is a particular case of Eq. (16) for $S = e^\Theta$, so Eq. (18) is also true. Further, with the help of Eq. (D9) we get: $\eta \Lambda(\theta) \eta = \Lambda^T(-\theta)$. On the other hand, it is obviously that $\Lambda(-\theta) = \Lambda^{-1}(\theta)$. Therefore, we have: $\Lambda^T(\theta) \eta \Lambda(\theta) = \eta$. This proves Eq. (20) for the case $\det(S) = +1$.

Let $\det(S) = -1$. Any matrix $S_- \in GL(4, M)$ with $\det(S_-) = -1$ can be represented in the form $S_- = S_+ S_P$ where $S_+ \in GL(4, M)$, $\det(S_+) = +1$,

$$S_P = \frac{1}{2} (1 - i \gamma^1 - \gamma^2 \gamma^5 - \gamma^3 \gamma^4). \quad (\text{D11})$$

The matrix $S_P \in GL(4, M)$. It has $\det(S_P) = -1$ and generates the space-time permutation. Namely, substituting S_P into Eq. (18) we obtain $\Lambda = \bar{I}_6$ with the matrix \bar{I}_6 defined in Eq. (9). Since $\bar{I}_6^T \eta \bar{I}_6 = -\eta$, the matrix Λ generated by S_- also satisfies Eq. (20).

The extension of the proof of Eq. (20) to the general case is obvious because any matrix $S \in GL(4, M)$ can be represented in one of the two forms, namely: $S = \lambda S_+$ or $S = \lambda S_-$ where $S_\pm \in GL(4, M)$, $\det(S_\pm) = \pm 1$, λ is a real number. Eq. (21) is proved analogously.

The following remark is related to the property of the mapping $SL(4, M) \rightarrow L_+^\uparrow(3, 3)$. Consider matrices of the form: $\Lambda_+ = e^{2\theta} e^{2\theta'} e^{2\theta''}$ where the 6×6 real matrices θ , θ' , and θ'' satisfy Eq. (D9). From the above derivation it follows that the matrices Λ_+ form a group L_{Λ_+} , and $SL(4, M)$ is mapped *onto* L_{Λ_+} . Obviously, the group L_{Λ_+} is connected and $L_{\Lambda_+} \subseteq L_+^\uparrow(3, 3)$. On the other hand, it is easy to see that $\Lambda L_{\Lambda_+} \Lambda^{-1} \subseteq L_{\Lambda_+}$ for all Λ in $L_+^\uparrow(3, 3)$. Since the group $L_+^\uparrow(3, 3)$ is simple, we obtain that $L_{\Lambda_+} = L_+^\uparrow(3, 3)$.

-
- [1] L. Parker, Phys. Rev. **188**, 2287 (1969).
 - [2] E. Recami and R. Mignani, Riv. Nuovo Cimento **4**, 209 (1974).
 - [3] P. Demers, Canad. J. Phys. **53**, 1687 (1975).
 - [4] R. Mignani and E. Recami, Lett. Nuovo Cimento **16**, 449 (1976).

- [5] G. D. Maccarrone and E. Recami, *Found. Phys.* **14**, 367 (1984).
- [6] E. A. B. Cole, *Nuovo Cimento* **40A**, 171 (1977).
- [7] E. A. B. Cole, *Nuovo Cimento* **44B**, 157 (1978).
- [8] E. A. B. Cole, *J. Phys. A: Math. Gen.* **13**, 109 (1980).
- [9] E. A. B. Cole and S. A. Buchanan, *J. Phys. A: Math. Gen.* **15**, L255 (1982).
- [10] V. S. Barashenkov and M. Z. Yur'iev, *Nuovo Cimento* **112B**, 17 (1997).
- [11] V. S. Barashenkov, A. B. Pestov, and M. Z. Yur'iev, *Gen. Relat. Grav.* **29**, 1345 (1997).
- [12] V. S. Barashenkov, *Found. Phys.* **28**, 471 (1998).
- [13] J. Strnad, *J. Phys. A: Math. Gen.* **13**, L389 (1980).
- [14] J. Strnad, *J. Phys. A: Math. Gen.* **14**, L433 (1981).
- [15] L. Marchildon, A. F. Antippa, and A. E. Everett, *Phys. Rev. D* **27**, 1740 (1983).
- [16] C. E. Patty and L. L. Smalley, *Phys. Rev. D* **32**, 891 (1985).
- [17] J. B. Boyling and E. A. B. Cole, *Int. J. Theor. Phys.* **32**, 801 (1993).
- [18] G. Bertone, D. Hooper, and J. Silk, *Phys. Rep.* **405**, 279 (2005).
- [19] J. M. Overduin and P. S. Wesson, *Phys. Rep.* **402**, 267 (2004).
- [20] F. R. Gantmacher, *The Theory of Matrices* (Chelsea Publishing Co., New York, 1959) [Russian original, 4th ed. (Nauka, Moscow, 1988)].
- [21] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1, *Foundations* (Cambridge University Press, Cambridge, 1995).
- [22] T. Eguchi, P. B. Gilkey, and A. J. Hanson, *Phys. Rep.* **66**, 213 (1980).
- [23] N. Rosen, *Found. Phys.* **12**, 213 (1982).
- [24] M. Israelit, *Found. Phys.* **29**, 1303 (1999).
- [25] C. Wetterich, *Nucl. Phys.* **B222**, 20 (1983).
- [26] J. M. Overduin and P. S. Wesson, *Phys. Rep.* **283**, 303 (1997).
- [27] S. Weinberg, *The Quantum Theory of Fields*, Vol. 2, *Modern Applications* (Cambridge University Press, Cambridge, 1996).
- [28] M. Fairbairn, A. C. Kraan, D. A. Milstead, T. Sjöstrand, P. Skands, and T. Sloan, *Phys. Rep.* **438**, 1 (2007).
- [29] L. J. Rosenberg and K. A. van Bibber, *Phys. Rep.* **325**, 1 (2000).
- [30] R. Foot, *Int. J. Mod. Phys. D* **13**, 2161 (2004).
- [31] Z. Berezhiani, *Int. J. Mod. Phys. A* **19**, 3775 (2004).
- [32] E. A. B. Cole, *Nuovo Cimento* **115B**, 1149 (2000).
- [33] F. del Aguila, M. Masip, and M. Pérez-Victoria, *Nucl. Phys.* **B456**, 531 (1995).
- [34] A. Leike, *Phys. Rep.* **317**, 143 (1999).
- [35] T. Appelquist, B. A. Dobrescu, and A. R. Hopper, *Phys. Rev. D* **68**, 035012 (2003).
- [36] B. A. Dobrescu, *Phys. Rev. Lett.* **94**, 151802 (2005).
- [37] C. Boehm and P. Fayet, *Nucl. Phys.* **B683**, 219 (2004).